We note that the integrands in Eq. (15) and Eq. (16) differ only by a factor of $4\omega_1/m$. We approximate this by 2 over the entire range of integration. To take some amount of the angular dependence introduced by the denominator of the integrand of Eqs. (15) and (16) we use a modulation factor of $1/(1-\frac{1}{4}A)^4$. Then the desired small-angle estimate is

$$f(\theta_{12})\approx\text{const}\frac{\sin^2\theta_{12}(1-2\cos\theta_{12}+2\cos^2\theta_{12})}{(1-\frac{1}{4}A)^4}. \quad (17)$$

4. EXPERIMENTAL CONSIDERATIONS

Many details are discussed in the review article by Deutsch.[4] The contribution of the $^3S_1\rightarrow 3\gamma$ decay to the angular distribution can be greatly suppressed by using NO for quenching. Furthermore, this contribution[5] has the characteristic feature that it takes on its maximum value when two photons come out nearly together, i.e., for $\theta_{12}=0°$. On the other hand, the distribution given by Eq. (17) is zero for $\theta_{12}=0°$. Therefore, the known effect of the allowed 3γ decay can be unambiguously subtracted away to obtain an estimate of the strength of any possible interaction given by Eq. (5).

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Elastic Scattering of Pseudoscalar Mesons and SU_n Symmetry*

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We derive the crossing matrix for scattering $X_n+X_n\rightarrow X_n+X_n$ of a multiplet of scalar mesons X_n which transform according to the regular representation of SU_n, for all n. For $n=3$, or octet symmetry, in the limit of neglecting inelastic coupled channels we find that, if the $J^P=1^-$ 8′ channel resonates, so also should the other two 1^- channels 10 and $\overline{10}$. These decuplets resonate at essentially the same mass as the 8′, barring what are probably small corrections from crossed 0^+ channels 8 and 27. Similarly, for general n, all $J^P=1^-$ channels should resonate together. Application of the Gell-Mann and Okubo mass-splitting formula to the degenerate 8′, 10, and $\overline{10}$ leaves the 10 and $\overline{10}$ in energy regions which have been explored in experiments, at least if one assumes the 725-MeV $(K\pi)$ resonance is a member of the decuplet. However, effects from coupled inelastic channels such as $X_n+V_n\rightarrow X_n+V_n$ (V_n is $J^P=1^-$ multiplet) may remove the mass degeneracy in the limit before symmetry-breaking effects are introduced. The crossing relations for $n=3$ and $V_3=8'$, 10, $\overline{10}$ are examined and shown to be consistent with this explanation. For $n=3$, the $X_3+X_3\rightarrow X_3+X_3$ crossing relations favor a pseudoscalar, rather than a scalar octet X_3.

I. INTRODUCTION

RECENTLY, two principles have been used with rather striking success to classify the multitude of newly discovered baryonic resonances.[1] These are (a) that the resonances fall into sets which form irreducible representations of SU_3, the group of 3×3 unimodular matrices[2]; (b) that the operator which gives the symmetry-breaking mass-splittings between the members of each set is proportional to the eighth component of unitary spin.[3] The latter principle will receive strong confirmation if two baryon resonances which it predicts, the $S=-2$, Ξ_γ at 1600 MeV and the $S=-3$, Ω_δ^- at 1676 MeV, are discovered.

In the present paper, these two ideas are joined to a third; namely, that strongly interacting particles, in virtue of crossing symmetry, are self-generating (the "bootstrap" philosophy of Chew and Frautschi[4]). These three are then applied to the low-energy vector mesons $V=(\rho,K^*,\bar{K}^*,\omega)$ considered as resonances in the scattering $XX\rightarrow XX$ of pseudoscalar mesons $X=(\pi,K,\bar{K},\eta)$. What is here done for SU_3 is the equivalent of the problem for SU_2, isotopic spin, where $X=(\pi^+,\pi^0,\pi^-)$ and $V=(\rho_+,\rho_0,\rho_-)$; and the V resonance is self-generating, since the force which produces V is the exchange of V.[5] The SU_3 case differs from the SU_2 in that the strength of the symmetry-breaking interactions make an answer derived in the limit of exact SU_3 symmetry

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[1] S. L. Glashow and A. H. Rosenfeld, Phys. Rev. Letters **10**, 192 (1963).

[2] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report CTSL-20, 1961 (unpublished); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961). For a fuller bibliography and a discussion placing the SU_3 scheme in a context with other group-theoretical possibilities, see the rapporteur's talk, given by B. d'Espagnat, in *Proceedings of the 1962 International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962).

[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 395 (1961).

[5] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961), and erratum *ibid.*, p. 268.

a first approximation only. However, it is possible that the application of rule (b) to such answers would always lead to a much better approximation. We investigate that possibility for the present case in Sec. III.

The present calculation, which employs the N/D method, encounters the usual divergence problems arising from present lack of knowledge about high-energy phenomena and necessitating introduction of a cutoff parameter. Further, left-hand singularities of the amplitude are approximated by the contributions from the crossed resonances only. Both these procedures are typical, and the heuristic arguments in support of them will not be repeated here. Further, right-hand inelastic singularities such as those arising from $XX\rightarrow VX$ are neglected; this point will be discussed further in the conclusion.

The self-generating approach has not been followed through to its ultimate conclusions, as existence of the octet X has simply been assumed. Strictly speaking, all "input" particles should also appear as "output" resonances or bound states. To accomplish this for X we would have to consider, along with $XX\rightarrow XX$, some process such as $XV\rightarrow XV$ which has a $J^P=0^-$ channel. However, we do enforce the requirement of the bootstrap philosophy that resonances and bound states be self-consistent as well as self-generating. That is, the values of coupling strengths and resonance positions used in the "input" (crossed) channels agree with those calculated from the widths and positions of the resonances subsequently found in the "output" (physical) channel.

Just as for SU_2 where the two π can combine into states $I=0$, 1, or 2, so for SU_3 the two X can combine to form six states with multiplicities 1, 8, 8′, 10, $\overline{10}$, and 27. Again, just as for SU_2, some of these multiplets (1,8,27) are even under interchange and form only even-l states, while the others (8′,10,$\overline{10}$) form only odd-l states. The multiplets 10 and $\overline{10}$ must have the same mass since $\overline{10}$ contains the antiparticles of 10; the other multiplets are all self-conjugate. The particles (ρ,K^*,$\bar{K}^*$,ω) are conventionally grouped to form an 8′, but all the channels, not only the 8′, are considered in the present paper.

X is taken to be an octet even though SU_3 admits lower dimensional irreducible representations, in particular, its fundamental representations 3 and $\bar{3}$, which might be considered more basic. The version of SU_3 symmetry which admits the 3 and $\bar{3}$ is designated the Sakata model, while the version which admits only octets and representations formed by direct products of octets is designated octet symmetry. Experimental evidence presented by d'Espagnat[2] favors the latter. The octet occupies a special position among representations of SU_3 since it is the regular representation; i.e., the generators of SU_3 (analogs of the J_i for SU_2) transform as an octet.

Finally, it is interesting to see if the bootstrap approach can shed some light on the nonoccurrence of symmetries higher than SU_3. That is, we can calculate scattering with X belonging to the regular representation of SU_n, $n>3$, and inquire whether the crossing matrix is such as to permit a self-consistent solution in this case also. Ideally the bootstrap principle would prohibit a self-consistent solution for the higher n groups.

II. CROSSING RELATIONS

Crossing relations for octet-octet scattering have been given by Cutkosky and Tarjanne.[6] The derivation in that paper is mathematically formidable, however, so that a rederivation is worthwhile.

The contribution to the scattering amplitude from a crossed state β in, say, the t channel, contains a projection operator $P_\beta(t)$ which must be expanded as a linear combination of projection operators $P_\alpha(s)$ for the physical s channels in order for the effect of the crossed force $(\beta;t)$ on scattering in $(\alpha;s)$ to be known. The crossing matrices C are the expansion coefficients

$$P_\beta(t)=\sum_\alpha C(t\beta;s\alpha)P_\alpha(s)\,, \tag{2.1}$$

$$P_\beta(u)=\sum_\alpha C(u\beta;s\alpha)P_\alpha(s)\,. \tag{2.2}$$

In order to find the operators P, we first represent each member of a multiplet by a component of a tensor.[7] All tensors of SU_3 are products of the two fundamental tensors, T_i and its complex conjugate T^i ($i=1$, 2, 3). The octet is a tensor $A_j{}^i$. For instance, the component $A_1{}^2$ transforms like T^2T_1. Since T_i transforms like $(pn\Lambda)$, T^2T_1 transforms like $(\bar{n}p)$ or π^+. The operator to annihilate an arbitrary initial state of two mesons is $A_j{}^iB_l{}^k$.

To construct those linear combinations of the $A_j{}^iB_l{}^k$ which annihilate states belonging to an irreducible representation of SU_3, we start with those tensors obtained by symmetrizing, antisymmetrizing, and contracting indices on the AB in all possible ways[7,8]:

$$\begin{aligned}
T_s{}^s&\sim+A_j{}^iB_l{}^k+A_j{}^kB_l{}^i+A_l{}^iB_j{}^k+A_l{}^kB_j{}^i\,,\\
T_A{}^s&\sim+\cdots+\cdots-\cdots-\cdots\,,\\
T_s{}^A&\sim+\cdots-\cdots+\cdots-\cdots\,,\\
T_A{}^A&\sim+\cdots-\cdots-\cdots+\cdots\,,\\
R_j{}^k(R'_j{}^k)&\sim A_j{}^iB_i{}^k+(-)A_i{}^kB_j{}^i\equiv(AB+(-)BA)_j{}^k\,.\\
I&\sim A_j{}^iB_i{}^j\equiv(AB)\,.
\end{aligned} \tag{2.3}$$

s and A stand for symmetric and antisymmetric; R (and R') for regular, since the regular representation

[6] R. E. Cutkosky, and P. Tarjanne (to be published).

[7] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. Lee, Rev. Mod. Phys. 34, 1 (1962).

[8] A discussion, somewhat more extended than that of Ref. 7, on the relationship between irreducibility of a tensor and behavior of its indices under permutation can be found in M. Hamermesh, *Group Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), Chap. 10.

TABLE I. Expansion of the s-channel projection operators $P_\beta(ABCD)$ for the representation β of multiplicity $N_\beta(n)$ in SU_n. $(ABCD)$ is $A_l{}^iB_i{}^jC_j{}^kD_k{}^l$, $(AB)_j{}^i$ is $A_j{}^kB_k{}^i$, and (AB) is $A_j{}^iB_i{}^j$.

$P_\beta=$	P_I	P_R	$P_{R'}$	P_{A^s}	P_{s^A}	P_{A^A}	P_{s^s}
$(AB)(CD)$	$1/(n^2-1)$	$-2/(n^2-4)$				$1/2(n-1)(n-2)$	$1/2(n+1)(n+2)$
$(AD)(BC)$				$-1/4$	$-1/4$	$1/4$	$1/4$
$(AC)(BD)$				$1/4$	$1/4$	$1/4$	$1/4$
$(AB+BA)_j{}^i(CD+DC)_i{}^j$		$n/2(n^2-4)$				$-1/4(n-2)$	$-1/4(n+2)$
$(AB-BA)_j{}^i(DC-CD)_i{}^j$			$1/2n$	$-1/4n$	$-1/4n$		
$(ADBC)-(ACBD)$				$1/4$	$-1/4$		
$(ADBC)+(ACBD)$						$-1/4$	$1/4$
$N_\beta(n)$	1	n^2-1	n^2-1	$(n^4-n^2)/4-n^2+1$	$(n^4-n^2)/4-n^2+1$	$(n^2-n)^2/4-n^2$	$(n^2+n)^2/4-n^2$
$N_\beta(3)$	1	8	8	10	10	$\cdots$	27
$N_\beta(2)$	1	$\cdots$	3	$\cdots$	$\cdots$	$\cdots$	5

has one upper and one lower index. From $T_s{}^s$, $T_s{}^A$, $T_A{}^s$, R, R', and I will be built, respectively, 27, 10, $\overline{10}$, 8, 8′, and 1. The tensor from $T_A{}^A$ vanishes identically, at least for SU_n, with $n<4$. There is one more building block, the constant tensor $\delta_i{}^j$, already used implicitly in contracting. Products involving it,

$$\delta_j{}^iR_l{}^k,\ \delta_j{}^iR'_l{}^k,\ \delta_j{}^i\delta_l{}^kI,\ \delta_j{}^iI\,, \tag{2.4}$$

are now to be added to the tensors (2.3) with such coefficients that those tensors become traceless. (The tensors $A_j{}^i$ and $B_j{}^i$ already have vanishing traces: $A_i{}^i=B_i{}^i=0$.) Consider, for example, $T_s{}^s$ [now written in full detail, with inclusion of normalizing constants and trace-removing terms suppressed in (2.1)],

$$\begin{aligned}T_s{}^s=b_1[A_j{}^iB_l{}^k-\tfrac{1}{3}\delta_j{}^i\delta_l{}^k(AB)]&\\ +b_2\delta_j{}^i[A_kB^l+B_kA^l-\tfrac{2}{3}\delta_k{}^l(AB)]&\\ +b_3\delta_j{}^i\delta_l{}^k(AB)+(i\leftrightarrow k)&\\ +(j\leftrightarrow l)+(i\leftrightarrow k,\ j\leftrightarrow l)\,.&\end{aligned} \tag{2.5}$$

The expression is grouped to have the simplest behavior under contractions. Its symmetries further simplify the calculation of the b_i. If the tensor is traceless with respect to contraction of one pair of indices, it is automatically so with respect to contraction of the other three pairs. Furthermore, the contraction of the terms abbreviated $(j\leftrightarrow l)+(i\leftrightarrow k,\ j\leftrightarrow l)$ need not be computed explicitly, since it equals the contraction of the first terms with A and B interchanged.

We compute the initial-state projection operators for arbitrary n, since the only difference between the calculation for SU_n and SU_3 is that tensor indices range from 1 to n instead of 1 to 3.

$$\begin{aligned}T_I&=[n^2-1]^{-1/2}(AB)\,,\\ T_R&=[\tfrac{1}{2}n/(n^2-4)]^{1/2}[A_jB^i+B_jA^i-(2/n)\delta_j{}^i(AB)]\\ &\qquad\equiv[\tfrac{1}{2}n/(n^2-4)]^{1/2}R_j{}^i\,,\\ T_R{}'&=[2n]^{-1/2}[A_jB^i-B_jA^i]\equiv[2n]^{-1/2}R'_j{}^i\,,\\ T_s{}^A&=\tfrac{1}{4}[A_j{}^iB_l{}^k+n^{-1}\delta_j{}^iR'_l{}^k]+(\text{symm.})\,,\\ T_A{}^s&=\tfrac{1}{4}[A_j{}^iB_l{}^k+n^{-1}\delta_j{}^iR'_l{}^k]+(\text{symm.})\,,\\ T_A{}^A&=\tfrac{1}{4}[A_j{}^iB_l{}^k+[n(n-1)]^{-1}\delta_j{}^i\delta_l{}^k(AB)\\ &\qquad+[n-2]^{-1}\delta_j{}^iR_l{}^k+(\text{symm.})]\,,\\ T_s{}^s&=\tfrac{1}{4}[A_j{}^iB_l{}^k-[n(n+1)]^{-1}\delta_j{}^i\delta_l{}^k(AB)\\ &\qquad-[n+2]^{-1}\delta_j{}^iR_l{}^k]+(\text{symm.})\,.\end{aligned} \tag{2.3′}$$

In each case (symm.) denotes the three sets of terms, identical with the first, except for a permutation of indices and, if necessary, an over-all sign change, which give the indices of the tensor the correct behavior under interchange.

The over-all normalization constant is fixed by imposing on an appropriate tensor component the requirement

$$\langle 0|TT^\dagger|0\rangle=1/p\,. \tag{2.6}$$

p is the number of terms in the sum defining the projection operator, Eq. (2.8) below, which are equal because they contain the component T or a component equal to T by symmetry. For example, p is 1 for T_I, T_R, $T_R{}'$; but if to calculate the normalization of $T_A{}^s$ one were to pick component $(T_A{}^s)_{12}{}^{33}$, then $p=2$. The two equal terms in the sum, Eq. (2.8), are the ones containing $(T_A{}^s)_{12}{}^{33}$ and $(T_A{}^s)_{21}{}^{33}$. The meson operators $A_j{}^i$ and $B_j{}^i$ have been normalized as

$$\begin{aligned}\langle 0|A_j{}^i(A_k{}^l)^\dagger|0\rangle&=\langle 0|A_j{}^iA_l{}^k|0\rangle\\ &=\delta_l{}^i\delta_j{}^k-n^{-1}\delta_j{}^i\delta_l{}^k\,,\end{aligned} \tag{2.7}$$

where the n^{-1} term insures that the traces do not contribute. Although the same normalization constant applies to all components of the tensor, the requirement (2.6) may be applied only to those components which are unrelated to the others by the equations which express the vanishing of traces. For example, in com-

TABLE II. Column 1 lists the invariants needed to expand the t-channel projection operators $P_\alpha(ACBD)$. Columns 2–8 in turn expand these invariants in terms of the $P_\beta(ABCD)$. Table II is a rearrangement of the set of equations which invert the equations of Table I. A seventh invariant, $(ABCD)-(ADCB)$ may be obtained from the operators of Eq. (2.9).

	P_I	P_R	$P_{R'}$	P_{s^A}	P_{A^s}	P_{A^A}	P_{s^s}
$(AB)(CD)$	n^2-1						
$(AD)(BC)$	1	1	-1	-1	-1	1	1
$(AC)(BD)$	1	1	1	1	1	1	1
$(AC+CA)(BD+DB)$	$2(n^2-2)/n$	$(n^2-8)/n$	n			-2	$+2$
$(AC-CA)(BD-DB)$	$-2n$	$-n$	$-n$			-2	$+2$
$ABCD+ADCB$	$2(n^2-1)/n$	$(n^2-4)/n$	$-n$				

TABLE III. Crossing matrix $C(t\beta;\, s\alpha)$ for (regular representation)-(regular representation) scattering in SU_n. Column α gives the effect on physical s-channel α of a resonance in the crossed t channel of row β.

	P_I	P_R	P_R'	$P_s{}^A$	$P_A{}^s$	$P_A{}^A$	$P_s{}^s$
P_I	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$	$\frac{1}{n^2-1}$
P_R	1	$\frac{n^2-12}{2(n^2-4)}$	$\frac{1}{2}$	$\frac{-2}{n^2-4}$	$\frac{-2}{n^2-4}$	$\frac{-1}{n-2}$	$\frac{+1}{n+2}$
P_R'	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{n}$	$-\frac{1}{n}$
$P_s{}^A$	$\frac{n^2-4}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{2-n}{4n}$	$\frac{2-n}{4n}$
$P_A{}^s$	$\frac{n^2-4}{4}$	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{2-n}{4n}$	$\frac{2-n}{4n}$
$P_A{}^A$	$\frac{n^2(n-3)}{4(n-1)}$	$\frac{-n^2(n-3)}{4(n-1)(n-2)}$	$\frac{n(n-3)}{4(n-1)}$	$\frac{-n(n-3)}{4(n-1)(n-2)}$	$\frac{-n(n-3)}{4(n-1)(n-2)}$	$\frac{n^2-n+2}{4(n-1)(n-2)}$	$\frac{n-3}{4(n-1)}$
$P_s{}^s$	$\frac{n^2(n+3)}{4(n+1)}$	$\frac{n^2(n+3)}{4(n+1)(n+2)}$	$\frac{-n(n+3)}{4(n+1)}$	$\frac{-n(n+3)}{4(n+1)(n+2)}$	$\frac{-n(n+3)}{4(n+1)(n+2)}$	$\frac{n+3}{4(n+1)}$	$\frac{n^2+n+2}{4(n+1)(n+2)}$

puting the normalization of $T_s{}^s$ one might substitute $(T_s{}^s)_{12}{}^{33}$ into (2.6), but not $(T_s{}^s)_{32}{}^{31}$. The expression for the norm of a general component of T is not Eq. (2.6), but a tensor equation like Eq. (2.7), with terms such as the n^{-1} term in (2.7) to insure that traces do not contribute. Picking a tensor component for which no upper index equals a lower index has the additional advantage that all terms proportional to $\delta_j{}^i$ vanish. For example, $(T_s{}^s)_{12}{}^{33}$, Eq. (2.5), reduces to four terms and almost at sight it can be seen that $b_1=\frac{1}{4}$.

The s-channel $(AB\rightarrow CD)$ projection operators are given by, for instance, for $P_s{}^s$,

$$P_s{}^s(ABCD)=\sum_{ijkl} T_s{}^s(AB)_{jl}{}^{ik}[T_s{}^s(CD)_{jl}{}^{ik}]^\dagger. \quad (2.8)$$

The corresponding t-channel operator is $P_s{}^s(ACBD)$. Again the symmetries and tracelessness of the tensors (2.3′) simplify the calculation (2.8); e.g., computing $P_s{}^s$ reduces to contracting the single term $C_i{}^jD_k{}^l$ with $T_s{}^s(AB)_{jl}{}^{ik}$. Had the t-channel operators been defined by $P_s{}^s(ACDB)$ (i.e., forward scattering is $A\rightarrow D$), then the operators odd under interchange of B and D, $P_A{}^s$, $P_s{}^A$, and P_R', and corresponding rows of the crossing matrix, would change sign. This sign change would be compensated for by a corresponding sign change in the spatial factor of the amplitude.

Projection operators for the order $ABCD$ (s channel) are given in Table I; t-channel operators follow by $ABCD\rightarrow ACBD$. After inverting the one set of equations (see Table II) and substituting into the other, one gets the crossing matrix of Table III.

Two projection operators, those for $R\rightarrow R'$ and $R'\rightarrow R$, or

$$(1/2[n^2-4]^{1/2})(AB\pm BA)_j{}^i(DC\mp CD)_i{}^j, \quad (2.9)$$

have been ignored because for identical scalar particles these transitions would violate angular-momentum conservation.

To check the calculation one may verify that the crossing matrix satisfies the column-sum rule

$$(n^2-1)\delta_{\alpha I}=\sum_\beta C'(\alpha t;\,\beta s)N_\beta \quad (2.10)$$

and the row-sum rule

$$(n^2-1)\delta_{\beta I}=\sum_\alpha C(\alpha t;\,\beta s). \quad (2.11)$$

C' is the crossing matrix with the signs of rows R', $T_A{}^s$, $T_s{}^A$ changed (i.e., for the order $ACDB$ instead of $ACBD$ in the t channel), and N_β is the multiplicity of the βth representation (see Table I). These rules may be derived more easily by considering the P_β, not as products of annihilation operators, but as quantities with four subscripts ***abcd*** each ranging from 1 to N_R.

$$P_\beta(ABCD)\rightarrow \sum_{i=1}^{N_\beta} S(\beta i;\, cd)S(\beta i;\, ab)^*, \quad (2.12)$$

where $S(\beta i;\, ab)$ is the generalization of the Clebsch-Gordan coefficients for SU_n,

$$|\beta i\rangle=\sum_{A,B=1}^{N_R} S(\beta i;\, ab)|ab\rangle. \quad (2.13)$$

Now, for instance, (2.10) follows from

$$\sum_\beta P_\beta(ABCD)=(AC)(DB)=(n^2-1)P_I(ACDB), \quad (2.14)$$

[the first equality states that the sum over projection operators gives the projection operator for no unitary spin change] and

$$\sum_{A,B} S(\beta i;\, ab)S(\gamma j;\, ab)^* = \delta_{\beta\gamma}\delta_{ij}. \qquad (2.15)$$

An additional check,

$$C^2 = 1, \qquad (2.16)$$

is derived by noting that $ABCD \to ACBD$ is a permutation of two letters, which is, therefore, its own inverse.

A final check, suggested by Glashow[9] for $n=3$, utilizes the group O_8 of orthogonal transformations on eight objects, O_8 contains SU_3. Octet scattering in O_8 requires three invariant amplitudes for representations of dimension 1, 27 (antisymmetric), and 35 (symmetric). On restriction to SU_3, the no longer irreducible 27 and 35 split into (8, 10, $\overline{10}$) and (8, 27), respectively. Conversely, by adding together the appropriate rows of the SU_3 crossing matrix, one should get back the O_8 crossing matrix. This would imply identical 8 and 27 columns, as well as identical 8′, 10, and $\overline{10}$ columns. In fact, this test works for general n, since O_8 may be replaced by O_m, where m is the dimension (n^2-1) of the regular representation of SU_n.

The exchange of a crossed multiplet β will be attractive if the Born term is positive above threshold. For the t channel, the Born term $B_\beta(t,s)$ for S- or P-wave exchange is,

$$B_\beta(t,s) = P_\beta(ACBD)f_\beta, \qquad (2.17)$$
$$= \left(\textstyle\sum_\alpha C(\beta t;\, \alpha s)P_\alpha(ABCD)\right)f_\beta.$$

When $P_\beta = P_R{}'$, $P_A{}^s, P_s{}^A$ then

$$f_\beta = \frac{4g_t{}^2\cos\theta_t}{m_\beta{}^2 - t}g_\beta{}^2 = \frac{(s-u)}{m_\beta{}^2-t}g_\beta{}^2, \qquad (2.18)$$

otherwise

$$f_\beta = \frac{m_\beta{}^2}{m_\beta{}^2-t}g_\beta{}^2. \qquad (2.19)$$

In both cases $C(\beta t;\, \alpha s) > 0$ means attraction in channel α.

The u-channel contribution follows from the above by exchanging C and D and need not be computed explicitly. In fact, when the projection

$$B_{\beta l}(s) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta_s)P_l(\cos\theta_s) \times [B_\beta(u,s) + B_\beta(t,s)] \qquad (2.20)$$

is taken, the $B_\beta(u,s)$ contribution either exactly doubles or exactly cancels the $B_\beta(t,s)$ contribution according as the channel α of Eq. (2.15) is or is not allowed by Bose statistics to have angular momentum l.

We note that a separate study of the $P_A{}^s(\overline{10})$ and $P_s{}^A(10)$ s channels is unnecessary since behavior of the one follows from that of the other by charge conjugation invariance. Therefore, we may drop the $P_A{}^s$ column of the crossing matrix; similarly, we may add together the $P_s{}^A$ and $P_A{}^s$ rows. Then the submatrix for the odd-l channels (8′, 10, $\overline{10}$) reduces to

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \qquad (2.21)$$

TABLE IV. Crossing matrix $C(u\beta;\, s\alpha)$ for $8+10 \to 8+10$ scattering in SU_3, multiplied by 120.

	P_8	P_{10}	P_{27}	P_{35}
P_8	24	−48	−16	+48
$P_{\overline{10}}$	−60	90	−10	30
P_{27}	−54	−27	111	27
$P_{\overline{35}}$	210	105	35	15

The result of an N/D bootstrap calculation for these channels follows immediately from the SU_2 result[5] for the $I=1$ channel since the $I=1$ diagonal element of the SU_2 crossing matrix is also a $\frac{1}{2}$: All three channels will resonate, and at the same mass.

The 8′ and (10, $\overline{10}$) channels will no longer be exactly degenerate once the effects of the crossed S-wave channels 8 and 27 are included; however, it is hard to imagine a self-consistent solution in which the 8 and 27 resonate strongly, since the net effect of crossed $8'+10+\overline{10}$ on the 8 and 27 is repulsive. The remaining S-wave channel, the unitary singlet 1, cannot remove the mass degeneracy since, as a crossed force, it contributes equally to 10, $\overline{10}$, and 8′.

Next to the crossed channels, inelastic channels offer the simplest mechanism to produce a mass-splitting. For example, XX could couple via $XX \to XV$ to $XV \to XV$, where V is 8′, 10, or $\overline{10}$. The net effect of the Born terms in $XV \to XV$ is unknown *a priori* because in most of these terms coupling constants occur linearly. Therefore, the crossing matrices for $XV \to XV$ cannot be used to predict in which directions the (10, $\overline{10}$) and 8′ masses will be shifted; however, they will show whether these directions are the same or opposite.

For V transforming as $T_{R'}$, the crossing matrix for t-channel exchange $C(t,s)$ is what has already been derived. Since the four external particles are no longer identical, the matrix $C(u,s)$ for u-channel exchange is now needed explicitly. $C(u,s)$ is the permutation $ABCD \to ADCD$, which equals the product of the three successive permuations $ABCD \to ABDC \to ADBC \to ADCB$. The first and third affect only the over-all signs of the projection operators and may, therefore, be summarized by a matrix σ which is a unit matrix except for −1's at the $P_s{}^A$, $P_A{}^s$, $P_R{}'$ diagonal positions. Then

$$C(u,s) = \sigma C(t,s)\sigma. \qquad (2.22)$$

The crossed 8, as well as 8′, u channel may now contain a $J^P = 1^-$ resonance, since for the inelastic amplitude the external particles are no longer identical.

[9] Sheldon Glashow (private communication).

From the crossing matrix it is seen that such a force could move $(10, \overline{10})$ and $8'$ masses in opposite directions.

For V transforming as $T_s{}^A$, we calculate only the case $n=3$. The general case would involve nine channels. For $n=3$ there are four, of multiplicity 8, 10, 27, 35 in the s channel. For $n=3$ only, $T_s{}^A$ is equivalent to a tensor V_{ijk} symmetric in all three lower indices. Its normalization is

$$\langle 0|\, V^{\alpha\beta\gamma}V_{ijk}\,|0\rangle = \tfrac{1}{6}(\delta_i{}^\alpha\delta_j{}^\beta\delta_k{}^\gamma + [\text{symmetrize with respect to } \alpha,\beta,\gamma]). \quad (2.23)$$

For $n=3$ only the completely antisymmetric constant tensor ϵ_{ijk} can be used to abbreviate the antisymmetrization of indices. The conventions $AV\to CU$, $AC\to VU$, and $AU\to CV$ are used to define s, t, and u channels, respectively. Tensors for the initial state, s channel, are

$$\begin{aligned}
(T_8)_\alpha{}^\beta &= (2/5)^{1/2}\epsilon^{\beta ij}V_{\alpha ik}A_j{}^k,\\
(T_{10})_{\alpha\beta\gamma} &= (1/12)^{1/2}\\
&\quad\times\{V_{\alpha\beta i}A_\gamma{}^i + (\alpha\leftrightarrow\gamma) + (\beta\leftrightarrow\gamma)\},\\
(T_{27})_{\alpha\beta}{}^{\mu\nu} &= (3/16)^{1/2}\{\epsilon^{\mu ij}V_{\alpha\beta i}A_j{}^\nu - (1/5)\\
&\quad\times[\delta_\beta{}^\nu\epsilon^{\mu ij}V_{\alpha ki}A_j{}^k + (\alpha\leftrightarrow\beta)]\\
&\quad + (\mu\leftrightarrow\nu)\}, \qquad (2.24)\\
(T_{35})_{\alpha\beta\gamma\nu}{}^\mu &= (1/4)\{V_{\alpha\beta\gamma}A_\nu{}^\mu - (1/6)\\
&\quad\times[V_{\alpha\beta i}A_\nu{}^i\delta_\gamma{}^\mu + (\alpha\leftrightarrow\gamma) + (\beta\leftrightarrow\gamma)]\\
&\quad + (\alpha\leftrightarrow\nu) + (\beta\leftrightarrow\nu) + (\gamma\leftrightarrow\nu)\}.
\end{aligned}$$

Tensors for the u channel, and for the final state in the s channel, follow from the above by relabeling and/or complex conjugation. Tensors for the initial state, t channel, have already been computed, at Eq. (2.3′); tensors for the final state are

$$\begin{aligned}
T_I &= (U^{ijk}V_{ijk})(\tfrac{1}{10})^{1/2} \equiv (\tfrac{1}{10})^{1/2}(UV),\\
(T_8)_\alpha{}^\beta &= (\tfrac{3}{5})^{1/2}[V_{\alpha ij}U^{\beta ij} - (1/3)\delta_\alpha{}^\beta(UV)],\\
(T_{27})_{\alpha\beta}{}^{\mu\nu} &= (18/7)^{1/2}\{V_{\alpha\beta i}U^{\mu\nu i} - (1/5)\\
&\quad\times[\delta_\alpha{}^\mu(V_{\beta ij}U^{\nu ij}) + (\alpha\leftrightarrow\beta) + (\mu\leftrightarrow\nu) \qquad (2.25)\\
&\quad + (\alpha\leftrightarrow\beta,\ \mu\leftrightarrow\nu)]\\
&\quad + (1/20)(\delta_\alpha{}^\mu\delta_\beta{}^\nu + \delta_\alpha{}^\nu\delta_\beta{}^\mu)(UV)\}.
\end{aligned}$$

The four u channels transform as 8, $\overline{10}$, 27, 35; the four t channels as 27, $8'$, 8, 1. The crossing matrices for $8+10\to 8+10$ (Tables IV, V) follow using the procedures outlined previously. When a channel is inelastic, as is the t channel in this case, the over-all phase of the corresponding projection operator is a matter of convention. Table V is defined for those t-channel operators which result from simply contracting the tensors (2.25) with the tensors (2.3′) (after relabeling the latter $AB\to AC$). Any Born term in an inelastic channel is proportional to the product of two coupling constants. Changing the phase of a t-channel projection operator amounts to redefining the relative phase between the corresponding pair of coupling constants.

TABLE V. Crossing matrix $C(t\beta; s\alpha)$ for $8+10\to 8+10$ scattering in SU_3, multiplied by $60\sqrt{5}$.

	P_8	P_{10}	P_{27}	P_{35}
P_I	15	15	15	15
P_8	24	36	-36	12
$P_8{}'$	$-60\sqrt{2}$	$-30\sqrt{2}$	$-10\sqrt{2}$	$30\sqrt{2}$
P_{27}	$-27\sqrt{70}$	$27\sqrt{70}$	$3\sqrt{70}$	$-(27/7)\sqrt{70}$

The Born terms for $8+\overline{10}\to 8+\overline{10}$ are the complex conjugates of those for $8+10\to 8+10$. Hence, the former reinforce the effect of the latter in the 8 (and $8'$) and 27 channels and the former affect the $\overline{10}$ and $\overline{35}$ as the latter affects the 10 and 35.

Again the situation exemplified by matrix (2.21) does not persist to the inelastic channels.

III. APPLICATION OF MASS-SPLITTING FORMULA

We compute the effect of mass splitting, transforming as the eighth component of unitary spin, on multiplets 8, 10, $\overline{10}$ which are degenerate, or nearly degenerate, in the limit of exact SU_3 symmetry. We wish to investigate whether the decuplets could be responsible for the departure of the vector octet masses from the sum rule predicted by the mass-splitting formula for an octet alone,

$$m_K{}^2 = \tfrac{1}{4}(3m_\omega{}^2 + m_\rho{}^2). \qquad (3.1)$$

Further, we wish to investigate where the other members of the decuplet would fall, according to the mass-splitting formula, if either the $I=\frac{1}{2}$ or $I=\frac{3}{2}$ component were identified with the recently discovered enhancement in $(K\pi)$ final states at 725 MeV.[10]

Just as for the j_z dependence of the matrix elements of J_1, J_2, and J_3, so also the isotopic spin and hypercharge dependence of the matrix elements of the mass tensor m^2 are determined by group theory. They vanish unless $\Delta Y=\Delta I^2=0$, in which case they are given by

$$\langle 8'|\,m^2\,|8'\rangle = a_1 + 2(I^2 - \tfrac{1}{4}Y^2)a_2 + a_6 Y, \qquad (3.2)$$

$$\langle 10|\,m^2\,|10\rangle = a_5 - a_3 Y, \qquad (3.3)$$

$$\langle 10|\,m^2\,|8'\rangle = [(5/8)Y^2 + (5/4)Y - (1/2)I^2]a_4. \qquad (3.4)$$

The matrix elements for $\overline{10}$ states follow from charge conjugation, under which $Y\to -Y$. When applied to Eq. (3.2), this operation yields

$$a_6 = 0. \qquad (3.5)$$

Equation (3.4) is needed only when the $8'$ and $(10, \overline{10})$ are similar in mass, so that states from each could couple in the SU_2 limit. The 10 contains states with quantum numbers $(|I|, Y) = (0,-2)$, $(\frac{1}{2},-1)$, $(1,0)$, $(\frac{3}{2},1)\equiv\lambda^-$, $\bar{K}_{10}{}^*$, ρ_{10}, $K_{3/2}{}^*$. The positions of both the ρ_8 and the $K_8{}^*$, of the octet, could be shifted by the

[10] S. G. Wojcicki, G. R. Kalbfleisch, and M. H. Alston, Bull. Am. Phys. Soc. 8, 341 (1963).

coupling, Eq. (3.4), to the corresponding particles in the 10 and $\overline{10}$ so that Eq. (3.1) would no longer apply.

In discussing the ρ_{10} and $\rho_{\overline{10}}$ it is more convenient to use linear combinations of definite G parity,

$$|\rho_\pm\rangle = (1/\sqrt{2})[|\rho_{10}\rangle \pm |\rho_{\overline{10}}\rangle]. \qquad (3.6)$$

Since $\rho_+(\rho_-)$ decays into only even (odd) numbers of pions, in the SU_2 limit only ρ_+ couples to ρ_8.

Computing from Eqs. (3.2)–(3.4) the masses m_ρ^2, m_K^2, etc., in terms of the a_i is a straightforward eigenvalue problem. When $a_4=0$, the result is a uniformly spaced decuplet with spacing a_3 (a familiar result, from the application of Eq. (3.2) to the higher baryons[1]); and an octet obeying equation (3.1) with a $K_8{}^*$ mass intermediate between ρ_8 and ω masses. When $a_4\neq 0$, $K_8{}^*$ and $K_{10}{}^*$ couple and "repulse" each other, that is, move in opposite directions away from $\frac{1}{2}(a_1+a_2+a_5+a_3)$, their mean position when $a_4=0$ is

$$m^2(K_{1/2}{}^*)\equiv m_{1/2}{}^2=\tfrac{1}{2}(a_1+a_2+a_5+a_3) \pm[\tfrac{1}{4}(a_1+a_2-a_5-a_3)^2+a_4{}^2]^{1/2}. \qquad (3.7)$$

Similarly, for ρ_8 and ρ_+,

$$m^2(\rho_8,\rho_+)=\tfrac{1}{2}(a_1+4a_2+a_5) \pm[\tfrac{1}{4}(a_1+4a_2-a_5)^2+2a_4{}^2]^{1/2}. \qquad (3.8)$$

The positions of the other resonances are as for $a_4=0$:

$$m^2(\omega)=a_1\,, \qquad (3.9)$$

$$m^2(\lambda)=a_5+2a_3\,, \qquad (3.10)$$

$$m^2(\rho_-)=a_5\,, \qquad (3.11)$$

$$m^2(K_{3/2}{}^*)=a_5-a_3\,. \qquad (3.12)$$

There are three well-established masses $m^2(\rho)$, $m^2(K^*)$, $m^2(\omega)$ and five constants a_i. Rather than discuss all possible solutions to Eqs. (3.2)–(3.4), we confine attention to those which give a $|Y|=1$ particle at 725 MeV. We will consider solutions for both assignments $I=\frac{1}{2}$ and $I=\frac{3}{2}$ for this particle, even though the latter value seems excluded at present by the work of Barbaro-Galtieri *et al.* on $(K^-\pi^-)$ final states.[11] Let $K^{*\prime}$ stand for the (mass)^2 of this particle, and similarly ρ, ω and K^* for the (mass)^2 of the other observed vector resonances.

Equations (3.7)–(3.8), squared and rewritten as equations for the unknowns a_i, become two two-sheeted hyperbolas in the (a_5,a_2) plane. The positions of their vertices and asymptotes depend upon a_3 and a_4 as parameters.

$$(a_5-\rho)[a_2-\tfrac{1}{4}(\rho-\omega)]=\tfrac{1}{2}a_4{}^2\,, \qquad (3.8')$$

$$[a_5-(a_3-m_{1/2}{}^2+2\rho)][a_2-(\tfrac{1}{2}\omega+\tfrac{1}{2}\rho-m_{1/2}{}^2)] =(-m_{1/2}{}^2+a_3+\rho)[(3\omega+\rho)\tfrac{1}{2}-2m_{1/2}{}^2]. \qquad (3.13)$$

Equation (3.13) is a combination of Eqs. (3.7) and (3.8) to eliminate a_4.

[11] A. Barbaro-Galtieri, A. Hussain, and R. D. Tripp, Bull. Am. Phys. Soc. 8, 326 (1963).

TABLE VI. Spectra calculated from the Gell-Mann Okubo mass-splitting formula. The row labeled I gives the isospin assumed for the $K^{*\prime}$. (Mass)^2 of the observed ρ, $K_{1/2}{}^*$, $K^{*\prime}$, and ω are 0.56, 0.78, 0.53, 0.62 (BeV)^2.

Resonance or parameter	(Mass)^2 or parameter value (BeV^2)				
I	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
a_3	0.10	+0.22	−0.20	−0.03	−0.03
a_5	0.63	0.56	0.88	0.56	0.66
$K_{3/2}{}^*$	0.53	0.34	1.08	0.59	0.69
ρ_+, ρ_8	0.80, 0.56	0.26, 0.56	0.98, 0.56	1.26, 0.56	0.96, 0.56
ρ_-	0.63	0.56	0.88	0.56	0.66
$K_{1/2}{}^*$, $K_{1/2}{}^{*\prime}$	0.78, 0.61	0.78, 0.53	0.78, 0.53	0.78, 0.53	0.78, 0.53
λ	0.83	1.00	0.48	0.50	0.60

Since $a_4{}^2\geqq 0$, hyperbola (3.8′) must lie in quadrants 1 or 3 with respect to its asymptotes, while hyperbola (3.13) is 2–4 or 1–3 according as $a_3\gtrless m_{1/2}{}^2-\rho$ [the other factor, $\frac{1}{2}(3\omega+\rho)-2m_{1/2}$, will always be negative in the analysis which follows].

Consider first the assignment $I=\frac{3}{2}$ for the $K^{*\prime}$. Then from Eq. (3.12),

$$K^{*\prime}=a_5-a_3 \quad (I=\tfrac{3}{2}\text{ only}). \qquad (3.14)$$

Also, $m_{1/2}{}^2=K^*$. For $a_3>K^*-\rho$ hyperbolas (3.8′), (3.13) cannot intersect except at points which violate the constraint (3.14). For $a_3<K^*-\rho$, intersection requires $a_2>\frac{1}{4}(\rho-\omega)$ or, substituting this in Eq. (3.13), $a_5<K^*-a_3$. Then using Eq. (3.14), we get an upper bound, $a_3<\frac{1}{2}(K^*-K^{*\prime})$. A lower bound $a_3>\rho-K^{*\prime}$ follows by eliminating a_2 and a_5 from (3.8′), (3.13), and (3.14); and imposing $a_4{}^2>0$. Therefore, a_3 is restricted to the range

$$+0.03\leqq a_3\leqq +0.125(\text{BeV})^2 \quad (I=\tfrac{3}{2}\text{ only}). \qquad (3.15)$$

The spectrum for $a_3=0.10(\text{BeV})^2$ is given in Table VI. As a_3 decreases to $+0.03$, $m^2(\lambda)$ and $m^2(\rho_-)$ decrease to ω and ρ, respectively. As a_3 increases to 0.125, $m^2(\lambda)$ increases to 0.91 (BeV)^2. Therefore, for $I=\frac{3}{2}$, λ would be stable with respect to strong interaction, since the threshold for decay to two K mesons is 0.976 (BeV)^2.

If $I=\frac{1}{2}$ for the $K^{*\prime}$, then Eq. (3.13) yields two hyperbolas when $m_{1/2}{}^2=K^*$ and $m_{1/2}{}^2=K^{*\prime}$ are substituted in turn. In place of the $m_{1/2}{}^2=K^{*\prime}$ hyperbola, however, one can use the simpler equation obtained from Eq. (3.7) by adding the two roots together:

$$K^*+K^{*\prime}=a_5+a_2+a_3+\omega \quad (I=\tfrac{1}{2}\text{ only}). \qquad (3.16)$$

From Eqs. (3.8′), (3.13) (with $m_{1/2}{}^2=K^*$), and (3.16) we get

$$\Delta^2+\Delta[\tfrac{1}{2}(3\omega+\rho)-(K^*+K^{*\prime})] -(K^{*\prime}-\rho-a_3)(K^*-\rho-a_3)=0\,, \quad (I=\tfrac{1}{2}\text{ only}). \qquad (3.17)$$

$$\Delta\equiv a_5-\rho\,.$$

The determinant of this quadratic has zeros at $a_3\approx 0$, $a_3\approx 0.20$, and there are no solutions for $0\lesssim a_3\lesssim 0.20$. Table VI gives spectra for $a_3=0.22$, -0.20, and -0.03. Spectra for $a_3>0.22$ have even lower values for $m^2(\rho_-)$.

For large values of $|a_3|$, $m^2(\lambda)\approx 0.61+a_3$. Near $a_3=0$ both roots of the quadratic (3.17) are admissible; otherwise one of the roots is always excluded by the requirement that a_4^2 calculated from Eq. (3.8′) be positive.

IV. CONCLUSION

The crossing relations for pseudoscalar meson scattering in the limit of exact SU_3 symmetry and the neglect of inelastic coupled channels, predicts the existence of 10 and $\overline{10}$ decuplets of vector mesons, in addition to the 8′ octuplet. The decuplets would have the same mass as the octuplet, or practically the same mass, neglecting what are probably minor corrections from the crossed 8 and 27 channels.

The hypothesis that the $|S|=1$, $I=\frac{1}{2}$ or $\frac{3}{2}$ members of the decuplet may be identified with the recently discovered 725-MeV $K\pi$ resonance is strongly rejected by experiments. If the 725-MeV resonance has $I=\frac{1}{2}$, then the mass-splitting formula predicts in this energy region $S=\pm 2$, singly charged $I=0$ bound states which have not been observed. If the 725-MeV resonance has $I=\frac{3}{2}$ those particles would be at much higher mass, but again, evidence is against any $|S|=1$, $I=\frac{3}{2}$ resonances below 1 BeV.[11] Further, the spin-parity of the $K^{*\prime}$ is not known, and may not be $J^P=1^-$ as required by this hypothesis.

Of course, it is entirely possible that the mass-splitting formula for some reason does not supply the proper symmetry-breaking corrections in the meson case. But such a conclusion is not at all required by the present investigation, because the limit of exact SU_3 symmetry has not been worked out in full detail but only in the absence of inelastic effects. A detailed calculation is needed to show whether such effects destroy the (10, $\overline{10}$), or simply split the (10, $\overline{10}$) from the 8′. The latter case is certainly allowed by present theoretical and experimental knowledge. If this splitting were great enough, the decuplets could be found in the symmetry-breaking limit as a group of resonances in the until now unexplored energy region above 1 BeV. A self-consistent, bootstrap calculation by Zachariasen and Zemach has shown that the mass of the ρ resonance in $\pi\pi$ scattering is nearly doubled by inclusion of the coupled inelastic $\pi\omega$ channel.[12]

If the decuplets were indeed found in this region, their existence would provide support for a conjecture due to Glashow,[9] that exact SU_3 symmetry is itself the limit obtained by breaking a still higher symmetry, that of O_8. Octets in O_8 scatter into states of multiplicity 1, 28, and 35; and on going to SU_3 the 28 splits into $8'+10+\overline{10}$.

[12] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

The best hypothesis to explain why the mass-splitting rule is so poorly obeyed by the vector meson octet remains that of Sakurai,[13] that the $I=0$ 1020-MeV $K\bar{K}$ resonance is a $J^P=1^-SU_3$ singlet which couples to the ω in the broken-symmetry limit and pushes it below the K^*. (In the absence of such a coupling the mass-splitting formula predicts the K^* should lie between ρ and ω.) The decuplets, if at higher energy, tend to depress the K^* rather than raise it above the ω.

The scattering amplitude for the regular representation of SU_n, $n>3$, was found to possess resonances analogous to the 8′, 10, and $\overline{10}$ of the $n=3$ case; i.e., an "SU_n universe" is quite possible, at least in the present simple approximation. The forces from these crossed $J^P=1^-$ particles into the $J^P=0^+$ unitary singlet channel are all attractive and increasing with n. Should the analogs of the 10 and $\overline{10}$ survive for higher n, then a low-energy resonance or bound state must inevitably result for all n above some minimum. This is the channel with the quantum numbers of the Pomeranchuk Regge pole, so that the asymptotic behavior of the amplitude would alter with n. In addition, the inelastic process $XX\rightarrow XS$, with S the scalar, should certainly be taken into account. Experimentally, a near-resonant interaction, the ABC, has been found in this channel.[14]

Nowhere in the calculation of Born terms or crossing relations have we made use of the intrinsic parity of the mesons X. If these were 0^+ instead of 0^-, however, then self-consistency would require (and conservation of J and P no longer forbid) a bound state of unit mass in the 8 channel together with a crossed force from the 8. This state is self-repulsive in its own channel, however. Thus, octet symmetry together with the bootstrap principle offers an argument (subject to the limitations regarding inelastic effects) for the pseudoscalarity of the pion.

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